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# Remarks on Hirota's $D$-operators, the double-complex function method and the Ernst equation

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**Abstract.** In this paper, a double matrix equation with Hirota's $D$-operators is given. Besides the original solutions available, this equation provides half the exact solution lost in the common Hirota direct method for solving the Ernst equation. Using this double matrix equation, we obtain a realization of the double Ehlers transformation group. In addition, we discuss the inverse problem about the equation.

## 1. Introduction

Due to the importance of the Ernst equation [1] in general relativity and Yang–Mills field theory, various schemes and techniques to find exact solutions have been discussed by a great many authors. Recently, Sasa and others [2–5] have suggested the use of the Hirota direct method [6] to solve the Ernst equation. In this paper we prove that in their schemes, in fact, half the exact solutions are lost. In order to obtain the total number of exact solutions, we suggest consideration of a double matrix equation with Hirota's $D$-operators, and we discuss the solutions and some applications of the equation.

In this paper we use the so-called double-complex function method. It is known that besides ordinary complex numbers, of which the imaginary unit is i ($\mathrm{i}^2 = -1$), there exist other generalized complex numbers [7], namely, the hyperbolic complex numbers (or double numbers), of which the imaginary unit is $\epsilon$, where $\epsilon^2 = +1$ and $\epsilon \neq \pm 1$. Corresponding to the ordinary complex number field $C$, all hyperbolic complexes $a + \epsilon b$ ($a$ and $b$ are real) constitute a commutative ring $H$. In the double-complex function method the ordinary complex numbers and the hyperbolic complex numbers are combined, the general method is detailed in [8]. Let $J$ denote the double imaginary unit, i.e. $J = \mathrm{i}$ or $J = \epsilon$. When all $a_n$'s are real numbers, and $\sum_{n=0}^{\infty} |a_n|$ is a convergent series, then the real number $a(J) = \sum_{n=0}^{\infty} a_n J^{2n}$ is called a double real number. It corresponds to a real number pair $(a_C, a_H)$, where $a_C = a(J = \mathrm{i}) = \sum_{n=0}^{\infty} (-1)^n a_n$ and $a_H = a(J = \epsilon) = \sum_{n=0}^{\infty} a_n$. If both $a(J)$ and $b(J)$ are double real numbers, we call $Z(J) = a(J) + Jb(J)$ a double-complex number. $Z(J)$ corresponds to a complex number pair $(Z_C, Z_H)$, where $Z_C = a_C + \mathrm{i} b_C$ and $Z_H = a_H + \epsilon b_H$. In this paper we employ only the double-complex functions with two real variables, for example $f(J) = f(x, y; J)$, $x, y \in R^1$, and let $f$ take the values in a set of double-complex numbers.

We call the following equation the double-complex Ernst equation:

$$\mathrm{Re}\,[\mathcal{E}(J)]\nabla^2\mathcal{E}(J) = \nabla\mathcal{E}(J) \cdot \nabla\mathcal{E}(J)$$

$$\nabla^2 = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2 \qquad \nabla = (\partial_\rho, \partial_z) \tag{1}$$

where $(\theta, \rho, z)$ is the usual cylindrical coordinate system, and $\mathcal{E}(J) = F(J) + J\Omega(J)$ is the double-complex Ernst potential, which is double-complex function of $\rho$ and $z$ only.

If $\mathcal{E}(J)$ is a solution of equation (1), then it corresponds to a complex Ernst potential pair $(\mathcal{E}_C, \mathcal{E}_H) = (F_C + \mathrm{i}\Omega_C, F_H + \epsilon\Omega_H)$. For details of the theories and applications of the double-complex Ernst equation, see [8–13]. In this paper we need to use the Neugebauer–Kramer transformation [14, 8], under which a hyperbolic complex Ernst potential can be changed into an ordinary complex Ernst potential as follows: if $\mathcal{E}_H = F_H + \epsilon\Omega_H$ is given, let

$$\hat{\mathcal{E}}_C = \hat{F}_C = \mathrm{i}\hat{\Omega}_C \qquad \hat{F}_C = \frac{\rho}{F_H} \qquad \partial_\rho\hat{\Omega}_C = \frac{\rho}{F_H^2}\partial_z\Omega_H \qquad \partial_z\hat{\Omega}_C = -\frac{\rho}{F_H^2}\partial_\rho\Omega_H \tag{2}$$

then $\hat{\mathcal{E}}_C$ is just an ordinary complex Ernst potential.

We denote the two-dimensional Laplace–Hirota operator

$$\mathcal{D}^2 = D_\rho^2 + \frac{1}{\rho}D_\rho + D_z^2 \tag{3}$$

where $D_\rho$ and $D_z$ are Hirota's $D$-operators with respect to $\rho$ and $z$, which are defined by [6] ($f$ and $g$ are functions of $\rho$ and $z$ only):

$$D_\rho^m D_z^n f \cdot g = (\partial_\rho - \partial_{\rho'})^m (\partial_z - \partial_{z'})^n f(\rho, z) g(\rho', z')\Big|_{\substack{\rho'=\rho \\ z'=z}} = (-1)^{m+n} D_\rho^m D_z^n g \cdot f. \tag{4}$$

Suppose that $\mathbf{M}(J)$ is a symmetric double $2 \times 2$ matrix with the form

$$\mathbf{M}(J) = \begin{bmatrix} M_{11}(J) & JM_{12}(J) \\ JM_{12}(J) & M_{22}(J) \end{bmatrix} \qquad (M_{11}(J) \neq 0). \tag{5}$$

In particular, the determinant of $\mathbf{M}(J)$ must be negative definite, i.e. $\det[\mathbf{M}(J)] = M_{11}(J)M_{22}(J) - J^2M_{12}^2(J) < 0$, where $M_{ij}(J) = M_{ij}(\rho, z; J)(i, j = 1, 2)$ are double-real functions of $\rho$ and $z$ only. We may as well denote $\det[\mathbf{M}(J)] = -P^2(J)$, where $P(J) = P(\rho, z; J) > 0$ is a positive definite double-real function of $\rho$ and $z$ only; therefore $\mathbf{M}(J)$ is

$$\mathbf{M}(J) = \begin{bmatrix} M_{11}(J) & JM_{12}(J) \\ JM_{12}(J) & M_{11}^{-1}(J)[J^2M_{12}^2(J) - P^2(J)] \end{bmatrix}. \tag{6}$$

Now we consider the following double matrix equation with Hirota's $D$-operators

$$\mathcal{D}^2\mathbf{M}(J) \cdot \sqrt{|\det[\mathbf{M}(J)]|} = 0 \qquad \text{i.e. } \mathcal{D}^2\mathbf{M}(J) \cdot P(J) = 0. \tag{7}$$

By transforming the dependent variable as

$$F(\rho, z; J) = \frac{P(\rho, z; J)}{M_{11}(\rho, z; J)} \qquad \Omega(\rho, z; J) = \frac{M_{12}(\rho, z; J)}{M_{11}(\rho, z; J)} \tag{8}$$

and developing equation (7), we obtain

$$\begin{aligned} F(J)\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right]F(J) &= [\partial_\rho F(J)]^2 + [\partial_z F(J)]^2 + J^2([\partial_\rho\Omega(J)]^2 + [\partial_z\Omega(J)]^2) \\ F(J)\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \partial_z^2\right]\Omega(J) &= 2[\partial_\rho F(J)\partial_\rho\Omega(J) + \partial_z F(J)\partial_z\Omega(J)]. \end{aligned} \tag{9}$$

We can easily verify that equation (9) is just the double-complex Ernst equation (1). This means that, by using equation (8), from each exact solution $\mathbf{M}(J)$ of (7) we can obtain a corresponding double-complex Ernst solution $\mathcal{E}(J)(M_{11}(J)$ must be non-vanishing). As for the inverse problem, we shall come back to it later.

When $J = \mathrm{i}$, equation (7) becomes

$$\mathcal{D}^2\mathbf{M}_C \cdot P_C = \mathcal{D}^2 \begin{bmatrix} M_{11C} & \mathrm{i}M_{12C} \\ \mathrm{i}M_{12C} & -M_{11C}^{-1}(M_{12C}^2 + P_C^2) \end{bmatrix} \cdot P_C = 0. \tag{10}$$

It is easily seen that equation (10) just leads to the cases discussed in [2–5]. From an exact solution $\mathbf{M}_C$ of equation (10) we obtain an ordinary complex Ernst potential

$$\mathcal{E}_C = F_C + \mathrm{i}\Omega_C = \frac{P_C}{M_{11C}} + \mathrm{i}\frac{M_{12C}}{M_{11C}}. \tag{11}$$

When $J = \epsilon$, equation (7) becomes

$$\mathcal{D}^2\mathbf{M}_H \cdot P_H = \mathcal{D}^2 \begin{bmatrix} M_{11H} & \epsilon M_{12H} \\ \epsilon M_{12H} & M_{11H}^{-1}(M_{12H}^2 - P_H^2) \end{bmatrix} \cdot P_H = 0. \tag{12}$$

This equation will give us those solutions lost in [2–5]. Corresponding to an exact solution $\mathbf{M}_H$ of equation (12), the hyperbolic complex Ernst potential is

$$\mathcal{E}_H = F_H + \epsilon\Omega_H = \frac{P_H}{M_{11H}} + \epsilon\frac{M_{12H}}{M_{11H}} \tag{13}$$

and by the Neugebauer–Kramer transformation (2), the corresponding ordinary complex Ernst potential is

$$\begin{aligned} &\hat{\mathcal{E}}_C = \hat{F}_C = \mathrm{i}\hat{\Omega}_C \qquad \hat{F}_C = \frac{\rho}{F_H} = \frac{\rho^M_{11H}}{P_H} \\ &\hat{\Omega}_C = \int \rho F_H^{-2}(\partial_z\Omega_H\,\mathrm{d}\rho - \partial_\rho\Omega_H\,\mathrm{d}z) = \int \frac{\rho M_{11H}^2}{P_H^2}\left[\partial_z\frac{M_{12H}}{M_{11H}}\mathrm{d}\rho - \partial_\rho\frac{M_{12H}}{M_{11H}}\,\mathrm{d}z\right]. \end{aligned} \tag{14}$$

In equation (14) the integrability condition has been guaranteed by (9). Generally $\hat{\mathcal{E}}_C \neq \mathcal{E}_C$, $\hat{\mathcal{E}}_C$ is a new solution. In addition, in general relativity we can obtain the axisymmetric gravitation solution directly from $\mathcal{E}_H$, in fact there is no need to use $\hat{\mathcal{E}}_C$ [8].

Some exact solutions of equation (10) (it corresponds to the common Ernst equation [1]) can often be extended easily to double solutions of equation (7). Sometimes, in some solutions of equation (10) we only need to substitute $J$ for i and make some simple changes; the results obtained are just the solutions of equation (7). As an example, we consider the Nakamura's solution series [3, 15]. To use the symbols in equation (10), the series is

$$P_C = \rho^{\bar{n}}A_C^{(n)} \qquad M_{11C} = \rho^{\bar{n}-n+1}A_C^{(n-1)} \qquad M_{12C} = \mathrm{i}\rho^{\bar{n}}\tilde{A}_C^{(n+1)} \tag{15}$$

where $\bar{n} = \frac{1}{2}n(n-2)$, $A_C^{(n)}$ and $\tilde{A}_C^{(n)}$, respectively, are determinants of two matrices whose entries are some monomials of the imaginary unit i and the real functions $u_{mC}(\rho, z)(m = 1, 2, 3, \ldots)$ [3]. Now, let $u_m(J) = u_m(\rho, z; J)$ be double-real functions determined by the following double recurrence relations:

$$\begin{aligned} &\left(\partial_\rho + \frac{m-1}{\rho}\right)u_m(J) = J^2\partial_z u_{m-1}(J) \\ &\left(\partial_\rho - \frac{m}{\rho}\right)u_{m-1}(J) = -J^2\partial_z u_m(J) \qquad (m = 1, 2, 3, \ldots). \end{aligned} \tag{16}$$

Substituting $J$ and $u_m(J)$ for i and $u_{mC}$ in the determinants $A_C^{(n)}$ and $\tilde{A}_C^{(n)}$, respectively, the results obtained read as

$$A^{(n)}(J) = \begin{Vmatrix} u_0(J) & Ju_1(J) & \ldots J^{n-1}u_{n-1}(J) \\ Ju_1(J) & u_0(J) & \ldots J^{n-2}u_{n-2}(J) \\ \vdots & \ddots & \vdots \\ J^{n-1}u_{n-1}(J) & J^{n-2}u_{n-2}(J) & \ldots u_0(J) \end{Vmatrix}$$

$$\tilde{A}^{(n)}(J) = \left\| \begin{array}{ccc} Ju_1(J) & u_0(j) & \ldots J^{n-3}u_{n-3}(J) \\ J^2u_2(J) & Ju_1(J) & \ldots J^{n-4}u_{n-4}(J) \\ \vdots & \ddots & \vdots \\ J^{n-1}u_{n-1}(J) & J^{n-2}u_{n-2}(J) & \ldots Ju_1(J) \end{array} \right\|. \tag{17}$$

Let equation (15) be extended to

$$P^{(n)}(J) = \rho^{\bar{n}} A^{(n)}(J) \qquad M_{11}^{(n)}(J) = \rho^{\bar{n}-n+1} A^{n-1}(J) \qquad M_{12}^{(n)}(J) = J\rho^{\bar{n}} \tilde{A}^{n+1}(J) \tag{18}$$

then it can be directly verified that $M^{(n)}(J)$ consisting of $M_{ij}^{(n)}(J)$ and $P^{(n)}(J)$ in equation (18), indeed, is a series of solutions of (7). When $J = \mathrm{i}$ equation (18) returns to (15), i.e. the Nakamura's solution series. When $J = \epsilon$, from equations (13) and (14) we obtain a new series of solutions of the ordinary complex Ernst equation as follows:

$$\hat{\mathcal{E}}_C^{(n)} = \frac{A_H^{(n-1)}}{\rho^{n-2}A_H^{(n)}} + \mathrm{i}\int \frac{\left[A_H^{(n-1)}\right]^2}{\rho^{2n-3}\left[A_H^{(n)}\right]^2} \left\{ \partial_z \frac{\rho^{n-1}\epsilon\tilde{A}_H^{(n+1)}}{A_H^{(n-1)}}\,\mathrm{d}\rho - \partial_\rho \frac{\rho^{n-1}\epsilon\tilde{A}_H^{(n+1)}}{A_H^{(n-1)}}\,\mathrm{d}z \right\}$$
$$(n = 1, 2, 3, \ldots). \tag{19}$$

These $\hat{\mathcal{E}}_C$'s cannot be obtained from the original Nakamura's solutions.

Using equation (7), we can easily obtain a realization of the double Ehlers transformation group [8]. Let $\pi(J)$ denote the set of all matrices with forms as

$$\mathbf{S}(J) = \begin{bmatrix} d(J) & Jc(J) \\ Jb(J) & a(J) \end{bmatrix} \qquad \det[\mathbf{S}(J)] = a(J)d(J) - J^2b(J)c(J) = 1 \tag{20}$$

where $a(J)$, $b(J)$, $c(J)$ and $d(J)$ are double-real constants. It is easily seen that $\pi(J)$ forms a group (in fact, it is a subgroup of the so-called $SU(2; J)$ group [8, 11]). Now, for any $\mathbf{S}(J)\epsilon\pi(J)$ we define a transformation $\mathcal{J}_S(J)$ acting upon matrix $\mathbf{M}(J)$ in equation (5) by

$$\mathcal{J}_s(J){:}\, \mathbf{M}(J) \to \tilde{\mathbf{M}}(J) = \mathbf{S}(J)\mathbf{M}(J)\mathbf{S}^t(J) \tag{21}$$

where $\mathbf{S}^t$ is the transpose of $\mathbf{S}$, then $\tilde{\mathbf{M}}(J)$ is still symmetric and its form is still (see the following equation (25))

$$\tilde{\mathbf{M}}(J) = \begin{bmatrix} \tilde{M}_{11}(J) & J\tilde{M}_{12}(J) \\ J\tilde{M}_{12}(J) & \tilde{M}_{11}^{-1}(J)[J^2\tilde{M}_{12}^2(J) - P^2(J)] \end{bmatrix}$$
$$\det[\tilde{\mathbf{M}}(J)] = \det[\mathbf{M}(J)] = -P^2(J). \tag{22}$$

Since $a(J)$, $b(J)$, $c(J)$ and $d(J)$ are constants, for any exact solution $\mathbf{M}(J)$ of equation (7) we have

$$\mathcal{D}^2\tilde{\mathbf{M}}(J) \cdot \sqrt{|\det[\tilde{\mathbf{M}}(J)]|} = \mathcal{D}^2\mathbf{S}(J)\mathbf{M}(J)\mathbf{S}^t(J) \cdot P(J)$$
$$= \mathbf{S}(J)[\mathcal{D}^2\mathbf{M}(J) \cdot P(J)]\mathbf{S}^t(J) = 0. \tag{23}$$

This means that $\tilde{\mathbf{M}}(J)$ is an exact solution of equation (7) and $\mathcal{J}_S(J)$ is a transformation generating new solutions. The exact form of $\tilde{\mathbf{M}}(J)$ is

$$\tilde{\mathbf{M}}(J) = \begin{bmatrix} d^2M_{11} + 2J^2cdM_{12} + J^2c^2R & J(bdM_{11} + adM_{12} + J^2bcM_{12} + acR) \\ J(bdM_{11} + adM_{12} + J^2bcM_{12} + acR) & J^2(b^2M_{11} + 2abM_{12} + J^2a^2R) \end{bmatrix}$$
$$R \equiv \frac{J^2M_{12}^2 - P^2}{M_{11}} \tag{24}$$

where we have denoted simply $a \equiv a(J)$, $b \equiv b(J)$, etc. From equation (24) we obtain the corresponding double-complex Ernst potential

$$\tilde{\mathcal{E}}(J) = \tilde{F}(J) + J\tilde{\Omega}(J)$$

$$\tilde{F}(J) = \frac{F(J)}{[d(J) + J^2 c(J)\Omega(J)]^2 - J^2 c^2(J) F^2(J)}$$
$$\tilde{\Omega}(J) = \frac{[a(J)\Omega(J) + b(J)][d(J) + J^2 c(J)\Omega(J)] - a(J)c(J)F^2(J)}{[d(J) + J^2 c(J)\Omega(J)]^2 - J^2 c^2(J) F^2(J)}. \quad (25)$$

This result is just identical with the double Ehlers transformation [8]

$$\mathcal{J}(J): \mathcal{E}(J) \to \tilde{\mathcal{E}}(J) = \frac{a(J)\mathcal{E}(J) + Jb(J)}{Jc(J)\mathcal{E}(J) + d(J)} \qquad a(J)d(J) - J^2 b(J)c(J) = 1. \quad (26)$$

It can be directly verified that the group $\pi(J)$ is isomorphic to the double Ehlers transformation group. Therefore, the set $\{\mathcal{J}_S(J)\}$ for all $\mathbf{S}(J)$'s in $\pi(J)$ evidently forms transformation group, and it can be regarded as a realization of the double Ehlers transformation group, and the space acted upon by $J$ is the set $\{\mathbf{M}(J)\}$ of all exact solutions of equation (7).

Finally, we consider the inverse problem about equation (7): How are we to obtain the exact solution $\mathbf{M}(J)$ of equation (7) corresponding to a known double-complex Ernst potential $\mathcal{E}(J) = F(J) + J\Omega(J)$? From equations (3) and (4) we have the following identical relation for the functions $f$, $g$ and $h$ of $\rho$, $z$,

$$\mathcal{D}^2 hf \cdot hg = h^2 \mathcal{D}^2 f \cdot g + 2[h\partial_\rho^2 h - (\partial_\rho h)^2 + h\partial_z^2 h - (\partial_z h)^2] fg. \quad (27)$$

If the solution $\mathbf{M}(J)$ of equation (7) corresponds to a known $\mathcal{E}(J) = F(J) + J\Omega(J)$, thus from equation (8) we can write

$$P(J) = M_{11}(J)F(J) \qquad \mathbf{M}(J) = M_{11}(J)\mu(J)$$
$$\mu(J) = \begin{bmatrix} 1 & J\Omega(J) \\ J\Omega(J) & J^2\Omega^2(J) - F^2(J) \end{bmatrix}. \quad (28)$$

Using the identical relation (27), we have

$$\mathcal{D}^2 \mathbf{M}(J) \cdot P(J) = M_{11}^2(J) \mathcal{D}^2 \mu(J) \cdot F(J) + 2\{M_{11}(J)\partial_\rho^2 M_{11}(J) - [\partial_\rho M_{11}(J)]^2$$
$$+ M_{11}(J)\partial_z^2 M_{11}(J) - [\partial_z M_{11}(J)]^2\}\mu(J)F(J). \quad (29)$$

From the left-upper entries of the matrices in equation (29) we obtain

$$\tilde{\nabla}^2 F(J) + 2F(J)(\partial_\rho^2 + \partial_z^2)\ln|M_{11}(J)| = 0 \qquad \tilde{\nabla}^2 = \partial_\rho^2 - \frac{1}{\rho}\partial_\rho + \partial_z^2. \quad (30)$$

This means that $M_{11}(J)$ is only determined by $F(J)$ (and some suitable boundary conditions). Suppose that $M_{11}(J)$ is in the form $M_{11}(J) = \mathrm{e}^{-\theta(J)}$, where $\theta(J)$ is an unknown double-real function of $\rho$ and $z$ only. According to equation (30), $\theta(J)$ must obey the equation

$$\partial_\rho^2 \theta(J) + \partial_z^2 \theta(J) = \frac{\tilde{\nabla}^2 F(J)}{2F(J)} \quad (31)$$

which seems to be the known two-dimensional Poisson's equation (however, $\rho$ and z are not the Descartes coordinates). If an exact solution $\theta(J)$ is given, then we obtain the solution of the above inverse problem, i.e.

$$\mathbf{M}(J) = \begin{bmatrix} \mathrm{e}^{-\theta(J)} & J\mathrm{e}^{-\theta(J)}\Omega(J) \\ J\mathrm{e}^{-\theta(J)}\Omega(J) & \mathrm{e}^{-\theta(J)}[J^2\Omega^2(J) - F^2(J)] \end{bmatrix}. \quad (32)$$

In addition, from equations (29) and (30) we obtain an interesting equivalent form of the double-complex Ernst equation (1) with Hirota's $D$-operators, i.e.

$$\mathcal{D}^2 \mu(J) \cdot F(J) = \mu(J)\tilde{\nabla}^2 F(J). \quad (33)$$

We shall discuss this equation and some applications elsewhere.

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